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SISTEMAS UNIDIMENSIONAIS

Em uma dimensão, todo campo de força independente do tempo é conservativo:

$$\oint_C F(x)\,dx = G(x_0) - G(x_0) = 0\,, \quad \forall\, C$$

$$\Rightarrow F(x) \text{ é conservativo.}$$

INTEGRABILIDADE EM UMA DIMENSÃO

$$E = \frac{1}{2} m \left(\frac{dx}{dt}\right)^2 + U(x)$$

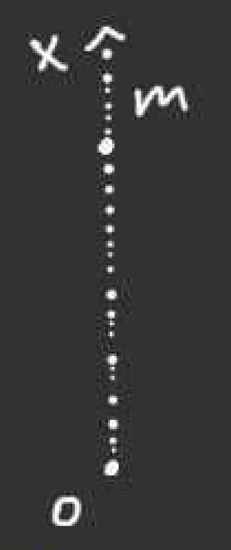
$$\Rightarrow \left(\frac{dx}{dt}\right)^2 = \frac{2}{m}\left[E - U(x)\right] \Rightarrow \frac{dx}{dt} = \pm \sqrt{\frac{2}{m}\left[E - U(x)\right]}\,.$$

$$\Rightarrow \int_0^t dt = \pm \int_{x(0)}^{x(t)} \frac{dx}{\sqrt{\frac{2}{m}(E - U)}} \Rightarrow \boxed{t = \pm \int_{x(0)}^{x(t)} \frac{dx}{\sqrt{\frac{2}{m}(E - U(x))}}}$$

$\downarrow$

Integrável por quadraturas.

Exemplo: Partícula de massa m, liberada do repouso de uma altura inicial x_0.
Atraída gravitacionalmente por $F = -\frac{K}{x^2}$, K constante. Qual x(t)?



$$t = -\int_{x_0}^{x(t)} \frac{dx}{\sqrt{\frac{2}{m}(E - U(x))}}$$

$$\Delta U = -\int_{x_i}^{x} F\,dx \Rightarrow U(x) - U(x_i) = \int_{x_i}^{x} \frac{K}{x^2}\,dx = -\frac{K}{x}\Big|_{x_i}^{x} = -K\left(\frac{1}{x} - \frac{1}{x_i}\right)$$

$$U(x_i \to \infty) = 0 \Rightarrow U(x) = -\frac{K}{x}. \qquad E = -\frac{K}{x_0}.$$

$$t = -\int_{x_0}^{x} \frac{dx}{\sqrt{\frac{2}{m}\left(-\frac{K}{x_0} + \frac{K}{x}\right)}} = -\sqrt{\frac{m}{2}}\sqrt{\frac{x_0}{K}} \int_{x_0}^{x} \frac{dx}{\sqrt{\frac{x_0}{x} - 1}}$$

$$\sqrt{\frac{x}{x_0}} = \operatorname{sen}\theta \Rightarrow \frac{x}{x_0} = \operatorname{sen}^2\theta \Rightarrow dx = x_0\, 2 \operatorname{sen}\theta \cos\theta\, d\theta$$

$$t = -\sqrt{\frac{m x_0}{2K}} \int \frac{2 x_0 \operatorname{sen}\theta \cos\theta\, d\theta}{\sqrt{\frac{1}{\operatorname{sen}^2\theta} - 1}} = -\sqrt{\frac{2 m x_0^3}{\cancel{2}K}} \int \frac{\operatorname{sen}\theta \cos\theta\, d\theta}{\sqrt{\frac{1 - \operatorname{sen}^2\theta}{\operatorname{sen}^2\theta}}}$$

$$= -\sqrt{\frac{2 m x_0^3}{K}} \int \frac{\operatorname{sen}\theta \cos\theta\, d\theta}{\sqrt{\frac{1}{\tan^2\theta}}} = -\sqrt{\frac{2 m x_0^3}{K}} \int_{\frac{\pi}{2}}^{\theta} |\tan\theta| \operatorname{sen}\theta \cos\theta\, d\theta$$

$$\operatorname{sen}\theta = \sqrt{\frac{x}{x_0}} : \sqrt{1} \to 0 = 1 \to 0 \Rightarrow \theta : \frac{\pi}{2} \to 0$$

$$\Rightarrow |\tan\theta| = \tan\theta$$

$$t = -\sqrt{\frac{2 m x_0^3}{K}} \int_{\frac{\pi}{2}}^{\theta} \operatorname{sen}^2\theta\, d\theta = -\sqrt{\frac{2 m x_0^3}{K}} \int_{\frac{\pi}{2}}^{\theta} \frac{1 - \cos 2\theta}{2}\, d\theta$$

$$= -\sqrt{\frac{2 m x_0^3}{K}} \cdot \frac{1}{2} \left[\theta - \frac{\pi}{2} - \frac{\operatorname{sen}(2\theta)}{2}\Big|_{\frac{\pi}{2}}^{\theta}\right]$$

$$= -\sqrt{\frac{m x_0^3}{2K}} \left[\theta - \frac{\pi}{2} - \frac{\operatorname{sen}(2\theta)}{2}\right] \qquad \theta = \operatorname{arcsen}\left(\sqrt{\frac{x}{x_0}}\right)$$

$$\boxed{t = -\sqrt{\frac{m x_0^3}{2K}} \left[\operatorname{arcsen}\left(\sqrt{\frac{x}{x_0}}\right) - \frac{\pi}{2} - \frac{1}{2}\operatorname{sen}\left(2 \operatorname{arcsen}\left(\sqrt{\frac{x}{x_0}}\right)\right)\right]}$$

Tempo de queda até a origem: $\tau = t(0) = \pi \sqrt{\frac{m x_0^3}{8K}}$.